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How far can a boundary coating material be carried downstream in turbulent pipe flow?

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Abstract. For a material that strongly attaches to the pipe wall the e-folding downstream attenuation distance is shown to be half the pipe radius multiplied by the square of the ratio between the bulk and friction velocities. This distance of a few hundred radii is significantly larger than the few tens of radii needed for cross-sectional mixing. Potential complexities such as departures from the von Kármán logarithmic velocity profile and shear dispersion are investigated but found to be relatively unimportant. Releasing the coating material at the centre of the flow, rather than uniformly across the flow, results in an uncoated entry region, but gives a one third increase in the concentrations far downstream.

1. Introduction

On a laboratory scale it is possible to coat the inside of an oil pipeline by advecting ultrafine particles of the coating material in a laminar flow along the pipeline. On a full scale with turbulent flows the process is ineffective – the coating only extends a few hundred pipeline radii downstream. A partial explanation for the difference is that the length scale for the coating material to reach and stick to the wall depends upon the diffusion across the pipe; in a laminar flow the diffusivity has tiny molecular values while in a turbulent flow the turbulent eddy diffusivity is several orders of magnitude larger. However, this partial explanation would suggest a penetration distance of an eddy decay scale of a few tens rather than a few hundreds of radii. The extra factor of ten is found to be associated with there being a diffusion boundary layer near the pipe wall. (For commercial reasons neither the laminar nor the turbulent experimental results were published).

2. Simple models for the flow and mixing

The mathematical literature on shear flow dispersion of heat or chemically reacting species has been focused principally upon laminar flows (e.g. Sankarasubramanian and Gill [1], Lungu and Moffatt [2]). The constant molecular diffusivities and simple velocity profiles have facilitated the derivation of neat expressions for the key bulk measures of the combined transport and loss process (the effective transport velocity and the loss rate). For turbulent open-channel flow with a parabolic eddy diffusivity and a von Kármán logarithmic velocity profile, Smith [3] and Purnama [4] have obtained similarly neat expressions for the effect of absorption at the bed. Such quasi-laminar treatments of turbulent flows rely on there being length or time scales well in excess of the decay scales for a turbulent eddy.

Taylor [5] showed that for turbulent pipe flow the departures from the von Kármán velocity profile had to be accounted for to get an accurate evaluation of the shear dispersion.

Accordingly, for the velocity profile w we include a function f which characterises the departure from the von Kármán wall layer:

$$w(\eta) = \bar{w}\{1 + \varepsilon W(\eta)\}\,, \tag{2.1a}$$

with

$$W(\eta) = \ln(1-\eta+\eta_*) + \frac{3}{2} + \eta_*(2+\eta_*)\ln\eta_* + \eta_* - (1+\eta_*)\ln(1+\eta_*) + \int_0^\eta \eta'^2 f\,\mathrm{d}\eta' - \int_\eta^1 (1-\eta'^2) f\,\mathrm{d}\eta'\,, \tag{2.1b}$$

$$\eta = r/a \quad \text{and} \quad \varepsilon = w_*/k\bar{w}\,. \tag{2.1c,d}$$

Here $\bar{w}$ is the cross-sectionally averaged velocity, w_* is the friction velocity (typically $\bar{w}/15$), k is von Kármán's constant (about 0.4), r is the radial coordinate and a is the radius of the pipe. The awkward constant terms involving η_*, ensure that the profile $W(\eta)$ gives zero contribution to the cross-sectionally averaged flow:

$$\int_0^1 2\eta W(\eta)\,\mathrm{d}\eta = 0\,. \tag{2.2}$$

To satisfy a no slip boundary condition on $r = a$ the function $W(\eta)$ has the large negative value $-1/\varepsilon$ on $\eta = 1$. Equivalently, the very small dimensionless roughness height η_* must have the value

$$\eta_* = \exp\left(-\frac{1}{\varepsilon} - \frac{3}{2} - \int_0^1 f\eta'^2\,\mathrm{d}\eta' - \eta_*(2+\eta_*)\ln\eta_* - \eta_* + (1+\eta_*)\ln(1+\eta_*)\right)\,. \tag{2.3}$$

Typically ε has the value $1/6$ and η_* the value 5×10^{-4}. Thus, the η_* constant terms in equation (2.1b) are numerically insignificant and can be ignored.

The family of velocity profiles (2.1a) can be associated with eddy viscosity profiles

$$v(\eta) = -\frac{kw_*a\eta}{\mathrm{d}W/\mathrm{d}\eta} = kw_*a\,\frac{\eta(1-\eta+\eta_*)}{1-(1-\eta+\eta_*)f} \tag{2.4}$$

(i.e. parabolic in the limiting case $f = 0$). Using Reynolds analogy, we shall use this same formula to determine the radial eddy diffusivity $\kappa(\eta)$. The concentration $c(r, z, t)$ of the coating material is assumed to satisfy the advection-diffusion equation

$$\partial_t c + w\partial_z c - \frac{1}{r}\,\partial_r(r\kappa\partial_r c) = 0\,, \tag{2.5}$$

where t denotes the time and z is the longitudinal position.

At the boundary there is deposition of the coating material. To allow for imperfect attachment we assume the boundary condition

$$\kappa\partial_r c + \beta c = 0 \quad \text{on} \quad r = a\,. \tag{2.6}$$

The bed absorption coefficient β characterises the strength of the initial attachment. There is no absorption when $\beta = 0$ and total retention when $\beta = \infty$.

At the discharge position $z = 0$ we assume that the rate of discharge $q(r, t)$ of the coating material is specified:

$$wc = q \quad \text{at} \quad z = 0 . \tag{2.7}$$

The discharge process is assumed to leave the velocity profile undisturbed. For the subsequent analysis it is convenient to assume that for times $t < 0$ the concentration c and the discharge rate q are identically zero.

Guided by the work of Smith [3] and Purnama [4] concerning open channel flow, we introduce the non-dimensional measures

$$Z = \left(\frac{w_*}{\bar{w}}\right)^2 \frac{z}{a}, \qquad T = \frac{w_*^2 t}{\bar{w} a}, \qquad B = \frac{\beta \bar{w}}{\beta \bar{w} + w_*^2}, \tag{2.8a,b,c}$$

of the longitudinal position, time and wall absorption. It is significant that the implicit length and time scales are long by a factor $\bar{w}/w_*$ compared to those of an individual turbulent eddy. Total retention ($\beta = \infty$) is now represented by $B = 1$. The equations satisfied by $c(\eta, Z, T)$ are

$$\varepsilon \frac{\partial c}{\partial T} + \varepsilon\{1 + \varepsilon W(\eta)\} \frac{\partial c}{\partial Z} - \frac{1}{\eta} \frac{\partial}{\partial \nu} \left(\frac{\eta^2 (1 - \eta + \eta_*)}{1 - (1 - \eta + \eta_*) f} \frac{\partial c}{\partial \eta} \right) = 0 , \tag{2.9a}$$

with

$$\frac{\eta_*}{1 + \eta_* f} \frac{\partial c}{\partial \eta} + \frac{\varepsilon B}{1 - B} c = 0 \quad \text{on} \quad \eta = 1 \tag{2.9b}$$

and

$$\{1 + \varepsilon W(\eta)\} \bar{w} c = q \quad \text{at} \quad Z = 0 . \tag{2.9c}$$

These equations involve just two key dimensionless parameters ε and B instead of the five parameters a, β, k, $\bar{w}$, w_* in the original formulation. It is elementary to revert from a solution $c(\eta, Z, T)$ in dimensionless variables to a solution $c(r, z, t)$ in conventional dimensional variables.

3. Spatial modes

To focus our attention upon the spatial structure along the pipeline we take Laplace transforms with respect to dimensionless time:

$$\hat{c} = \int_0^\infty \exp(-pT) c \, \mathrm{d}T . \tag{3.1}$$

The advection-diffusion equation, boundary and entrance conditions transform to

$$\varepsilon p \hat{c} + \left(\frac{1}{\varepsilon} + W(\eta) \right) \frac{\partial \hat{c}}{\partial Z} - \frac{1}{\eta} \frac{\partial}{\partial \eta} \left(\frac{\eta^2 (1 - \eta + \eta_*)}{1 - (1 - \eta + \eta_*) f} \frac{\partial \hat{c}}{\partial \eta} \right) = 0 , \tag{3.2a}$$

with

$$\frac{\eta_*}{1+\eta_* f}\frac{\partial \hat{c}}{\partial \eta}+\frac{\varepsilon B}{1-B}\hat{c}=0 \quad \text{on} \quad \eta=1\,, \tag{3.2b}$$

and

$$(1+\varepsilon W)\bar{w}\hat{c}=\hat{q} \quad \text{at} \quad z=0\,. \tag{3.2c}$$

On the axis $\eta=0$ the concentration c and the Laplace transform $\hat{c}$ are assumed to be nonsingular.

A separation of variables solution for $\hat{c}$ is

$$\bar{w}\hat{c}=\sum_{n=0}^{\infty}\Psi_n(\eta;\,p)\exp(-\mu_n Z)\int_0^1 2\eta'\hat{q}\Psi_n(\eta';\,p)\,\mathrm{d}\eta'\,, \tag{3.3}$$

The eigenmodes $\Psi_n(\eta;\,p)$ together with their eigenvalues $\mu_n(p)$ satisfy the Sturm–Liouville equations

$$\frac{\mathrm{d}}{\mathrm{d}\eta}\left(\frac{\eta^2(1-\eta+\eta_*)}{1-(1-\eta+\eta_*)f}\frac{\mathrm{d}\Psi_n}{\mathrm{d}\eta}\right)+\varepsilon(\mu_n(1+\varepsilon W)-p)\eta\Psi_n=0\,, \tag{3.4a}$$

with

$$\frac{\eta_*}{1+\eta_* f}\frac{\mathrm{d}\Psi_n}{\mathrm{d}\eta}+\frac{\varepsilon B}{1-B}\Psi_n=0 \quad \text{on} \quad \eta=1\,, \tag{3.4b}$$

and

$$\int_0^1 2\eta(1+\varepsilon W)\Psi_n^2\,\mathrm{d}\eta=1 \tag{3.4c}$$

It is conventional to index the modes with μ_n increasing

$$\mu_0<\mu_1<\cdots\,. \tag{3.4d}$$

Near the discharge position $z=0$ many modes may contribute to $\hat{c}$, depending upon the shape of the discharge. However, the combined effects of diffusion across the pipeline and absorption at the wall rapidly attenuate the higher modes. The eventual penetration of coating material along the pipeline is determined by just the zero mode contribution

$$\bar{w}\hat{c}\sim\Psi_0(\eta;\,p)\exp(-\mu_0 Z)\int_0^1 2\eta'\hat{q}\Psi_0(\eta';\,p)\,\mathrm{d}\eta'\,. \tag{3.5}$$

To determine the time dependence of the concentration $c(\eta, Z, T)$ far from the discharge position, we need to perform a Laplace inversion of equation (3.5).

In the triple limit $B=0$, $f=0$, $W=0$ the eigenmodes are Jacobi polynomials (Abramowitz and Stegun [6], chapter 22)

$$\Psi_n(\eta;\,p)=(n+1)^{1/2}P_n^{(0,1)}(2\eta-1)\,, \tag{3.6a}$$

$$\mu_n(p) = \frac{n(n+2)}{\varepsilon} + p \,. \tag{3.6b}$$

Hence, we can estimate that the e-folding length for the $n = 1$ mode to decay relative to the $n = 0$ mode is $\varepsilon/3$. In dimensional terms this mixing length is

$$\frac{\bar{w}a}{3kw_*} \tag{3.7}$$

i.e. about twelve radii downstream of the discharge position.

4. Factoring out the wall layer

In the limit as $\varepsilon = (w_*/k\bar{w})$ tends to zero the differential equation (3.4a) and the normalisation (3.4c) are satisfied by the rudimentary approximation

$$\Psi_0^{(0)} = 1 \,. \tag{4.1}$$

However, this approximation does not satisfy the absorption boundary condition (3.4b). A matched asymptotic expansion reveals that there is a wall layer of size η_* in which Ψ_0 exhibits behaviour similar to that of the velocity profile (2.1a,b). It is this diffusion boundary layer that prolongs the penetration process by a factor ε^{-1}. Guided by the work of Smith [3] and of Purnama [4], we represent the zero mode Ψ_0 in terms of the velocity profile $W(\eta)$:

$$\Psi_0(\eta; p) = [1 + \varepsilon BW(\eta)]\Phi_0(\eta; p) \,. \tag{4.2}$$

The definitions (2.1b), (2.3) of W and η_* imply that on $\eta = 1$ the square bracket has the value $(1 - B)$.

The Sturm–Liouville equations satisfied by the modified eigenmode Φ_0 and the eigenvalue μ_0 are:

$$\frac{\mathrm{d}}{\mathrm{d}\eta}\left(\frac{\eta^2(1-\eta+\eta_*)}{1-(1-\eta+\eta_*)f}[1+\varepsilon BW]^2\frac{\mathrm{d}}{\mathrm{d}\eta}\Phi_0\right) + \varepsilon(\mu_0 - 2B - p)(1+\varepsilon W)[1+\varepsilon BW]^2\eta\Phi_0$$
$$+ \varepsilon^2 2B[1+\varepsilon BW]\{1+B+\varepsilon BW\}W\eta\Phi_0 + \varepsilon^2 p[1+\varepsilon BW]^2 W\eta\Phi_0 = 0 \,, \tag{4.3a}$$

with

$$\frac{\mathrm{d}\Phi_0}{\mathrm{d}\eta} = 0 \quad \text{on} \quad \eta = 1 \,, \tag{4.3b}$$

and

$$\int_0^1 2\eta(1+\varepsilon W)[1+\varepsilon BW]^2\Phi_0^2 \,\mathrm{d}\eta = 1 \,. \tag{4.3c}$$

On the axis $\eta = 0$ the modified eigenfunction Φ_0 is assumed to be non-singular.

Integration of equation (4.3a) from $\eta = 0$ to $\eta = 1$ yields a formula for the eigenvalue $\mu_0(p)$:

$$(\mu_0 - 2B - p)\int_0^1 2\eta(1+\varepsilon W)[1+\varepsilon BW]^2\Phi_0\,\mathrm{d}\eta$$
$$= -\varepsilon 2B\int_0^1 2\eta[1+\varepsilon BW]\{1+B+\varepsilon BW\}W\Phi_0\,\mathrm{d}\eta - \varepsilon p\int_0^1 2\eta[1+\varepsilon BW]^2W\Phi_0\,\mathrm{d}\eta\,. \quad (4.4)$$

This formula will be used to obtain approximations to μ_0 from approximations for Φ_0.

5. Asymptotic expansions

In the limit as $\varepsilon = (w_*/k\bar{w})$ tends to zero all the equations (4.3a,b,c) are satisfied by the approximation

$$\Phi_0^{(0)} = 1\,. \quad (5.1)$$

There is now no need for matched asymptotic expansions. Any wall layer has been explicitly accounted for in the factorisation (4.2). From the integral formula (4.4) we can evaluate the first approximation to the eigenvalue

$$\mu_0^{(0)} = 2B + p\,. \quad (5.2)$$

The neat form of this result confirms the appropriateness of the nonlinear definition (2.8c) for the dimensionless boundary absorption coefficient B. It deserves emphasis that equations (5.1) and (5.2) do not involve the velocity profile W, so are not influenced by any departure from the von Kármán velocity profile.

Equations (4.3) and (4.4) have been written in a way which emphasises that corrections to Φ_0 and μ_0 do not arise until second order in terms of the small parameter ε:

$$\Phi_0 = 1 + \varepsilon^2\Phi_0^{(2)}(\eta; p) + \cdots\,, \quad (5.3a)$$

$$\mu_0 = 2B\{1-\varepsilon^2(2B+B^2)I\} + p\{1-\varepsilon^2 2BI\} + \varepsilon^3\mu_0^{(3)}(p) + \cdots\,, \quad (5.3b)$$

where

$$I = \int_0^1 2\eta W^2\,\mathrm{d}\eta\,. \quad (5.3c)$$

We note that if $f = 0$ then

$$I = \frac{5}{4}\,. \quad (5.4)$$

The integrand $2\eta W^2$ is largest near the wall where the von Kármán profile is most accurate. Hence, deviations from the von Kármán profile will not effect I significantly.

If we neglect $\Phi_0^{(2)}$, $\mu_0^{(3)}$ and higher order terms, then the Laplace transform solution (3.5) becomes

$$\bar{w}\hat{c} = [1 + \varepsilon BW(\eta)] \exp(-2B\{1 - \varepsilon^2(2B + B^2)I\}Z - p\{1 - \varepsilon^2 2BI\}Z)$$
$$\times \int_0^1 2\eta'[1 + BW(\eta')]\hat{q}\, \mathrm{d}\eta' . \tag{5.5}$$

The Laplace inversion can be performed exactly:

$$\bar{w}c(\eta, Z, T) = [1 + \varepsilon BW(\eta)] \exp(-2B\{1 - \varepsilon^2(2B + B^2)I\}Z)$$
$$\times \int_0^1 2\eta'[1 + \varepsilon BW(\eta')]q(\eta', T - \{1 - \varepsilon^2 2BI\}Z)\, \mathrm{d}\eta' . \tag{5.6}$$

Thus, the concentration signal associated with a time dependent discharge attenuates exponentially and travels at constant speed along the pipeline. The distribution across the flow of the suspended material exhibits a wall-layer structure, diminishing by a factor $(1 - B)$ at the wall. The effective source strength involves a weighted average across the flow, with the identical wall layer weighting. A discharge at the wall is a factor $(1 - B)$ weaker in its far field influence than a uniform discharge (i.e. a fraction B deposits on the wall close to the entry before equilibration across the flow has been achieved).

6. Attenuation along the pipeline

To convert the solution (5.6) into conventional dimensional variables we introduce the attenuation rate $\Lambda_0(\varepsilon)$ and the effective transport velocity $w_0(\varepsilon)$:

$$\Lambda_0 = 2\frac{B}{a}\left(\frac{w_*}{\bar{w}}\right)^2\{1 - \varepsilon^2(2B + B^2)I + \cdots\}, \tag{6.1a}$$

$$w_0 = \bar{w}\{1 - \varepsilon^2 2BI + \cdots\}^{-1} . \tag{6.1b}$$

The solution for the concentration far from the discharge position is

$$\bar{w}c(\eta, z, t) = [1 + \varepsilon BW(\eta)] \exp(-\Lambda_0 z) \int_0^1 2\eta'[1 + \varepsilon BW(\eta')]q\left(\eta', t - \frac{z}{w_0}\right) \mathrm{d}\eta' , \tag{6.2a}$$

For the cross-sectionally averaged concentration $\bar{c}$ or a uniform rate of discharge $q = \bar{q}(t)$, there are minor simplifications

$$\bar{w}\bar{c}(z, t) = \exp(-\Lambda_0 z) \int_0^1 2\eta'[1 + \varepsilon BW(\eta')]q\left(\eta', t - \frac{z}{w_0}\right) \mathrm{d}\eta' , \tag{6.2b}$$

$$\bar{w}c(\eta, z, t) = [1 + \varepsilon BW(\eta)] \exp(-\Lambda_0 z)\bar{q}\left(t - \frac{z}{w_0}\right) , \tag{6.2c}$$

The e-folding distance for the attenuation of the amount of coating material remaining in the flow is

$$\frac{1}{\Lambda_0} = \frac{a}{2B}\left(\frac{\bar{w}}{w_*}\right)^2\{1 + \varepsilon^2(2B + B^2)I + \cdots\}$$
$$= \frac{a}{2}\left(\frac{\bar{w}}{w_*}\right)^2\left[1 + \frac{w_*^2}{\bar{w}\beta} + \varepsilon^2 I\left(2 + \frac{\beta\bar{w}}{2(\beta\bar{w} + w_*^2)}\right) + \cdots\right] \tag{6.3}$$

i.e. about a hundred radii downstream. Unless there is perfect initial attachment ($\beta = \infty$),

the penetration can be increased by increasing the flow speed $\bar{w}$ (to speeds of order two hundred times β).

It is a well-established feature of laminar shear dispersion with boundary absorption that the loss rate asymptotes to a constant value as β tends to infinity (Sankarasubramanian and Gill [1] Fig. 2; Lungu and Moffatt [2] Fig. 1). The distinctive features of turbulent flows are the simple β dependence (Smith [3], Purnama [4]), the weakened dependence upon $\bar{w}$ and the much reduced penetration (a hundred rather than tens of thousands of radii downstream).

For the effective transport velocity w_0 the removal of slow moving particles near the wall has the effect of increasing the averaged speed experienced by the remaining particles. The wall layer character of the depletion process in turbulent flows implies that relatively few particles are involved and the speeding up is very slight. For example, in the double limit $f = 0$, $\beta = \infty$ the fractional increase in speed is only $\varepsilon^2 5/2$ i.e. about 0.07. By contrast, for laminar pipe flow the fractional increase in speed when $\beta = \infty$ is 0.564 (Gill and Sankarasubramanian [1], Lungu and Moffatt [2]).

7. Second-order corrections

The order ε^2 terms in equations (4.3a–c) can be satisfied:

$$\Phi^{(2)}(\eta; p) = -B\left(1 + \frac{1}{2}B\right)I + \{2B(1+B) + p\}g(\eta) \tag{7.1}$$

where the centroid displacement function $g(\eta)$ satisfies the equations

$$\frac{\mathrm{d}}{\mathrm{d}\eta}\left(\frac{\eta^2(1-\eta+\eta_*)}{1-(1-\eta+\eta_*)f}\frac{\mathrm{d}}{\mathrm{d}\eta}g\right) = -\eta W\,, \tag{7.2a}$$

with

$$\frac{\mathrm{d}g}{\mathrm{d}\eta} = 0 \quad \text{on} \quad \eta = 1\,, \tag{7.2b}$$

and

$$\int_0^1 2\eta g\,\mathrm{d}\eta = 0\,. \tag{7.2c}$$

The function g characterises the interaction between velocity shear and mixing. g is positive on the axis $\eta = 0$ and decreases to the negative value $-I/2$ on the wall $\eta = 1$. Since g does not exhibit a wall layer, we can neglect η_* in both sides of equation (7.2a). If $f = 0$ then we can use Jacobi polynomials to represent g:

$$g(\eta) = -2\sum_{n=1}^{\infty}\frac{(n+1)}{n^2(n+2)^2}P_n^{(0,1)}(2\eta - 1) \tag{7.3a}$$

with

$$g(0) = \frac{\pi^2}{12} - \frac{1}{8} = 0.697\,, \qquad g(1) = -\frac{5}{8} = -0.675\,. \tag{7.3b,c}$$

Deviations from the von Kármán velocity profile would effect the size and shape of $g(\eta)$ (i.e. the function f occurs explicitly on the left hand side of equation (7.2a) and implicitly on the right-hand side).

Correct to order ε^2 the solution for the concentration far along the pipe can be written

$$\begin{aligned}\bar{w}c(\eta, z, t) = {} & [1 + \varepsilon BW(\eta)]\{1 + \varepsilon^2 2B(1+B)g(\eta)\} \exp(-\Lambda_0 z - \varepsilon^2 B(2+B)I) \\ & \times \int_0^1 2\eta'[1 + \varepsilon BW(\eta')]\{1 + \varepsilon^2 2B(1+B)g(\eta')\} \\ & \times q\left(\eta', t - \frac{z}{w_0} + [g(\eta) + g(\eta')]\frac{a}{k^2\bar{w}}\right) \mathrm{d}\eta'\,.\end{aligned} \tag{7.4}$$

The changes from equation (6.2a) are: i) a perturbed distribution of particles across the pipe involving absorption, shear and mixing; ii) an enhanced loss of particles which we can attribute to entry effects while the depleted concentration wall layer is established; iii) time displacements which allow for the reduced (increased) time lapse when observations or releases are made in a fast (slow) part of the flow.

The best that can be done to get coating material far along the pipeline is to make the release on the axis $\eta' = 0$. For a von Kármán velocity profile we can use the results (2.1b, 7.3b, 7.4) to assess that the effective source strength is enhanced by the factor

$$\left[1 + \varepsilon\frac{3}{2}B\right]\left\{1 + \varepsilon^2 2B(1+B)\left(\frac{\pi^2}{12} - \frac{1}{8}\right) + \cdots\right\} \tag{7.5}$$

relative to a uniform discharge.

8. Shear dispersion

From equation (4.4) the second-order approximation for Φ_0 yields a third-order result for the eigenvalue $\mu_0(p)$:

$$\mu_0^{(3)} = B^2(2B+p)J - (2B(1+B)+p)^2 K\,, \tag{8.1a}$$

with

$$J = -\int_0^1 2\eta W^3\,\mathrm{d}\eta\,, \qquad K = \int_0^1 2\eta W g\,\mathrm{d}\eta\,. \tag{8.1b,c}$$

We record that if $f = 0$ then

$$J = \frac{7}{4}\,, \qquad K = 4\sum_{n=1}^{\infty}\frac{(n+1)}{n^3(n+2)^3} = \zeta(3) - \frac{7}{8} = 0.327\,. \tag{8.2a,b}$$

(The explicit ζ-function formula for K was derived by H.K. Kuiken). Deviations from the Kármán velocity profile would effect J very little but effect K substantially (because of much weaker importance of the near-wall contribution where W is large negative).

The constant terms in equation (8.1a) yield a third-order correction to the decay rate along the pipe

$$\Lambda_0 = 2\frac{B}{a}\left(\frac{w_*}{\bar{w}}\right)^2\{1-\varepsilon^2(2+B)BI+\varepsilon^3(B^2J-2B(1+B)^2K)+\cdots\}\,, \tag{8.3a}$$

Sensitivity to the velocity profiles arises principally at order ε^2 via the K coefficient. Similarly, the p terms in equation (8.1a) extend the series expansion for the transport velocity (with sensitivity at order ε^2 to the velocity profile):

$$w_0 = \bar{w}\{1-\varepsilon^2 2BI+\varepsilon^3(B^2J-4B(1+B)K)+\cdots\}^{-1}\,. \tag{8.3b}$$

The p^2 term in equation (8.1a) corresponds to shear dispersion with effective longitudinal diffusivity

$$D = K\frac{aw_*}{k^3}\,. \tag{8.4}$$

This result is independent of the absorption coefficient B. The numerical factor K as given in equation (8.2b) is low by about a factor of 2. Taylor [5] gave a numerical quadrature incorporating the experimental departure from the von Kármán velocity profile and giving near perfect agreement with his experimental results for the shear coefficient

$$D = 0.64\frac{aw_*}{k^3}\,. \tag{8.5}$$

In a diffusion type model for the coating process the discharge strength q in the solution (7.4) for $c(\eta, z, t)$ would need to be replaced by the smoothed (mollified) source strength

$$\int_0^\infty G\left(\tau, z-[g(\eta)+g(\eta')]\frac{a}{k^2}\right)q(\eta', t-\tau)\,\mathrm{d}\tau\,, \tag{8.6a}$$

with

$$G(\tau,\zeta) = \frac{w_0}{(\pi D\tau)^{1/2}}\exp\left(-\frac{(\zeta-\mathrm{w}_0\tau)^2}{4\mathrm{D}\tau}\right)-\frac{\mathrm{w}_0^2}{2\mathrm{D}}\exp\left(\frac{\zeta w_0}{D}\right)\mathrm{erfc}\left(\frac{\zeta+w_0\tau}{(4D\tau)^{1/2}}\right). \tag{8.6b}$$

The mollifier G is the solution of a diffusion equation with a sudden release at $\tau = 0$, $\zeta = 0$. On the length and time scale (2.8a,b) of the coating process, the mollifier G can be approximated by a Gaussian with respect to time:

$$G(\tau,\zeta) = \frac{1}{(2\pi\sigma_T^2)^{1/2}}\exp\left(-\frac{(\tau-\zeta/w_0)^2}{2\sigma_T^2}\right)\quad\text{with}\quad \sigma_T^2 = 2\frac{D\zeta}{w_0^3}\,. \tag{8.7a,b}$$

The temporal smoothing associated with the shear dispersion is unimportant unless the discharge varies on a time scale comparable with (or less than) the temporal spread σ_T^2.

9. Illustrative example

As a quantitative example we specify the dimensional values

$$a = 0.2\,\mathrm{m}\,,\qquad \bar{w} = 0.08\,\mathrm{m\,s^{-1}}\,,\qquad w_* = 0.0053\,\mathrm{m\,s^{-1}}\,. \tag{9.1}$$

(an oil pipeline with a flow of 10 litres per second). For the additional dimensionless parameters we take

$$k = 0.4\,, \quad \beta = \infty\,, \quad (B = 1)\,, \quad I = 1.25\,, \quad J = 1.75\,, \quad K = 0.64\,. \tag{9.2}$$

Thus, the expansion parameter ε has the value

$$\varepsilon = \frac{w_*}{k\bar{w}} = \frac{1}{6}\,. \tag{9.3}$$

The series expansion (8.3a) for the attenuation rate along the pipeline is

$$\Lambda_0 = \{1 - 0.104 - 0.015 + \cdots\}0.044m^{-1} = 0.039m^{-1} = \frac{1}{25.5m}\,. \tag{9.4}$$

The size of the terms to diminish reasonably rapidly. As noted earlier, the e-folding distance is about a hundred pipe radii.

The series (8.3b) for the effective transport velocity is

$$w_0 = \{1 - 0.069 - 0.015 + \cdots\}^{-1}0.08\ \mathrm{m\,s^{-1}} = 0.0873\ \mathrm{m\,s^{-1}}\,. \tag{9.5}$$

Again there is reasonably rapid decrease in the terms. The speed w_0 for the coating material is slightly larger than the mean velocity $\bar{w}$ of the bulk fluid. The time to traverse one e-folding distance is about 290 seconds.

In the second-order version (7.4) of the solution for the concentration, the time displacements at the axis $\eta = 0$ and at the wall $\eta = 1$ are

$$\frac{g(0)a}{k^2\bar{w}} = 10.9\ \mathrm{s}\,, \qquad \frac{g(1)a}{k^2\bar{w}} = -10.5\ \mathrm{s} \tag{9.6a,b}$$

Thus, the concentration signal arrives very slightly (of order ε^2) early in the faster central part of the flow and late in the slower moving fluid near the wall.

Taylor's [5] formula (8.5) for the shear dispersion coefficient gives

$$D = 0.011\ \mathrm{m^2\,s^{-1}} \tag{9.7}$$

Thus, over a distance of one e-folding, the temporal spread σ_T grows to

$$\sigma_T = \left(\frac{2D}{\Lambda_0 w_0^3}\right)^{1/2} = 29\ \mathrm{s}\,. \tag{9.8}$$

We can ignore shear dispersion if the release of coating material takes place on a longer time scale than this.

If we envisage the coating process as having a time scale comparable with the travel time for one e-folding (290 seconds), then neither the time displacements (9.6a,b) nor the temporal spread are significant. It deserves comment that, relative to the travel time, σ_T is of order $\varepsilon^{3/2}$ and is therefore larger than the time displacements (9.6a,b). However, for a smoothly varying rate of discharge the temporal smoothing has a quadratic effect of order σ_T^2, so is even less important than the displacements.

In this illustrative example of perfect boundary absorption it would be pointless to make a wall release, since none of the coating material would bet beyond the discharge position.

From equation (7.5) we find that relative to a uniform release the effective far field strength of a centre line release is enhanced by a factor

$$[1 + 0.25]\{1 + 0.077 + \cdots\} \sim 1.34 . \tag{9.9}$$

The price paid for this enhanced penetration would be an initial region of one mixing length

$$\frac{a\bar{w}}{3kw_*} = 2.5m \tag{9.10}$$

before much coating material reached the wall.

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